Comparison among several planar Fisher-KPP road-field systems

Andrea Tellini

1 Introduction

Road-field models are systems of reaction-diffusion equations posed in different spatial dimensions that have been introduced in the context of mathematical biology in [4] in order to take into account the effect that a line of fast diffusion has on the propagation in a half-plane, where a logistic-type reaction takes place.

More precisely, in [4], the authors consider a density $v(x,y,t)$ that diffuses in the upper half-plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$, called the *field*, with diffusion coefficient $d > 0$, and reproduces according to a reaction term $f(v)$, which is assumed to be of Fisher-KPP type. On the so-called *road*, i.e. the boundary of the half-plane given by $\{(x,y) \in \mathbb{R}^2 : y = 0\}$, another density $u(x,t)$ diffuses with a possibly different coefficient $D > 0$. In addition, a symmetric exchange between the road and the field is considered, with a fraction νv that passes from the field to the road and a fraction μu that, vice-versa, passes from the road to the field (μ, ν being positive constants). The corresponding reaction-diffusion system thus reads

$$\begin{cases} v_t - d\,\Delta v = f(v) & \text{for } (x,y,t) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ u_t - D\,u_{xx} = \nu\, v(x,0^+,t) - \mu\, u & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ -d\, v_y(x,0^+,t) = \mu\, u - \nu\, v(x,0^+,t) & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{cases} \tag{1}$$

where $\mathbb{R}^+$ denotes the set of positive numbers, $f : [0,\infty) \to \mathbb{R}$ is a Lipschitz function which is differentiable in 0 and satisfies

$$f(0) = f(1) = 0, \quad f > 0 \text{ in } (0,1), \quad f < 0 \text{ in } (1,\infty), \quad f(s) \le f'(0)s \text{ for } s \in [0,\infty), \tag{KPP}$$

and $v(x,0^+,t) := \lim_{y\downarrow 0} v(x,y,t)$, $v_y(x,0^+,t) := \lim_{y\downarrow 0} v_y(x,y,t)$.

Andrea Tellini
Universidad Autónoma de Madrid, Departamento de Matemáticas, Campus de Cantoblanco, 28049 Madrid (Spain), e-mail: andrea.tellini@uam.es

The study of such a system is motivated by many situations in nature in which some species or diseases spread faster along transportation networks (roads, rivers, railways) than in the surrounding environment. Some specific examples are the spreading of *Vespa velutina* in France (see [18]) or the early spread of HIV in the Democratic Republic of Congo (see [12]).

In [4], it has been proved that there exists a quantity, which will be denoted by c^*_{hp}, such that the solution of (1) starting from every continuous, compactly supported, nonnegative pairs $(u_0, v_0) \neq (0,0)$ (throughout all this work we will consider such a kind of initial data), converges to the unique positive steady-state of the system, $\left(\frac{\nu}{\mu}, 1\right)$, with an asymptotic speed of propagation in the direction of the road equal to c^*_{hp} (observe that the subindex refers to the domain, which is a half-plane).

By *asymptotic speed of propagation* in the direction of the road, i.e. the x direction, we mean that c^*_{hp} satisfies the following two properties:

(i) for all $c > c^*_{\text{hp}}$, $\lim_{t\to\infty} \sup_{\substack{|x|>ct \\ y\geq 0}} (u,v) = (0,0)$,

(ii) for all $a > 0$ and $c < c^*_{\text{hp}}$, $\lim_{t\to\infty} \sup_{\substack{|x|<ct \\ 0\leq y<a}} \left| (u,v) - \left(\frac{\nu}{\mu}, 1\right) \right| = 0$.

Such properties say that, asymptotically in time, the solution of the parabolic problem is close to the positive steady-state *inside* bounded rectangles expanding in the x direction at a speed *smaller* than c^*_{hp}, while it is still close to $(0,0)$ *outside* half-strips which are unbounded in y and expand in the in the x direction at a speed *larger* than c^*_{hp}.

The main result of [4] is a precise geometrical characterization of c^*_{hp} that, in particular, allows the authors to compare it with the speed of propagation of the Fisher-KPP equation, i.e. the first equation in (1), which is given by $c_{\text{KPP}} := 2\sqrt{df'(0)}$ (see, e.g., [1, 13, 15]). The results of [4] are summarized in the following theorem.

Theorem 1 ([4]). *Problem* (1) *admits an asymptotic speed of propagation in the x direction which will be denoted by c^*_{hp} and satisfies:*

(i) $c^*_{\text{hp}} \geq c_{\text{KPP}}$*;*
(ii) $c^*_{\text{hp}} > c_{\text{KPP}}$ *if and only if* $D > 2d$*;*
(iii) $\lim_{D\to\infty} c^*_{\text{hp}}(D) = \infty$.

In particular, these results establish that the speed of propagation can never be smaller than the one of a homogeneous environment and that the road enhances such a speed if and only if the diffusion D on it is larger than a certain threshold given by $2d$. Finally, this enhancement can be made arbitrarily large, by taking a sufficiently large D.

Several works on road-field systems in a half-plane have been carried out afterwards, with the goal of ascertaining more features of these models: in [5] additional reaction and transport terms have been considered on the road, in [6] the asymptotic speed of propagation in every direction has been determined, in [7] the existence of traveling fronts has been investigated, in [2, 3] a nonlocal diffusion is taken into account on the road, in [16, 17] nonlocal exchange terms and the relation between such a model and (1) are considered, in [14] μ and ν are allowed to depend periodically on x. A work that treats more general fields, which nonetheless are still unbounded in every direction, is [11], where the case of asymptotically conic domains is studied.

Other works devoted to road-field systems are related to fields with bounded section in the y direction: in [19] the analogue of system (1) is studied in the case of a strip-shaped field bounded by two roads with on which the diffusion is different with respect to the one in the field. Such a situation reads

$$\begin{cases} v_t - d\,\Delta v = f(v) & \text{for } (x,y,t) \in \mathbb{R} \times (-R,R) \times \mathbb{R}^+, \\ u_t - D\,u_{xx} = \nu\, v(x, \pm R^{\mp}, t) - \mu\, u & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \pm d\, v_y(x, \pm R^{\mp}, t) = \mu\, u - \nu\, v(x, \pm R^{\mp}, t) & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{cases} \tag{2}$$

as well as the corresponding higher-dimensional case, while in [20] the model with a strip bounded by only one road and homogeneous Dirichlet boundary conditions on the other line is handled; finally, [8, 9, 10] deal with (2) in the case of ignition-type reactions f.

The main result of [19] is the existence of an asymptotic speed of propagation, denoted by c^*_{st} (in order to refer to the strip-shaped field), in the x direction, which satisfies the properties summarized in the following theorem.

Theorem 2 ([19]). *Problem* (2) *admits an asymptotic speed of propagation* c^*_{st} *in the x direction which, in addition, satisfies:*

(i) $\lim_{R\downarrow 0} c^*_{\mathrm{st}}(R) = 0$*;*
(ii) $\lim_{R\to\infty} c^*_{\mathrm{st}}(R) = c^*_{\mathrm{hp}}$*;*
(iii) if $D \le 2d$*, the function* $R \mapsto c^*_{\mathrm{st}}(R)$ *is continuous and increasing;*
(iv) if $D > 2d$*, the function* $R \mapsto c^*_{\mathrm{st}}(R)$ *is continuous, and it is increasing for* $R \in (0, R_M)$ *and decreasing for* $R \in (R_M, \infty)$*, where* $R_M := \frac{\nu}{\mu}\frac{D}{D-2d}$.
Moreover, in this case, there exist $R_{\mathrm{hp}} \in (0, R_M)$ *and* $R_K \in (0, R_{\mathrm{hp}})$ *such that* $c^*_{\mathrm{st}}(R) > c^*_{\mathrm{hp}}$ *if and only if* $R > R_{\mathrm{hp}}$*, and* $c^*_{\mathrm{st}}(R) > c_{\mathrm{KPP}}$ *if and only if* $R > R_K$.

Observe that property (i) in Theorem 2 is new with respect to problem (1), whose speed of propagation is bounded away from 0, while (ii) can be seen as a continuous dependence result of the speed of propagation with respect to the domain. Indeed, one can think as one road in (2) to be fixed and, as $R \to \infty$, the other one lying further and further; thus the latter looses its effects on the propagation, and we recover problem (1).

Another similarity with Theorem 1 is the appearance of the same threshold $2d$ for the diffusion D, but now related to the monotonicity of c^*_{st} with respect to the size of the strip. As remarked in [19], the emergence of two types of monotonicity can be explained by the lack of reaction on the road: if $D \le 2d$ it is more convenient for the population to propagate in the interior of the strip, where both the reaction and the diffusion are better than on the boundary; thus a larger strip makes the speed of propagation larger. On the contrary, if $D > 2d$, on the one hand it is better to have a larger field for the effect of the reaction to be greater, but, on the other hand, the roads are now more convenient for the diffusion and, by increasing R, they become further apart. The competition between these effects, entails the existence of an optimal distance of the roads which maximizes the speed of propagation.

By comparing Theorems 1 and 2, it is apparent that road-field systems may behave in an extremely different way according to whether the section of the field is bounded or not. In this work we pursue this study by analyzing the combined effect of a part of field with bounded width together with another one with unbounded width, the two parts being separated by two roads where the diffusion is different with respect to the one in the field.

With respect to (1) and (2), observe that we have to allow two-side exchanges; for this reason, we first generalize the analysis of [4] to the system

$$\begin{cases} v_t - d\,\Delta v = f(v) & \text{for } (x,y,t) \in \mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^+, \\ u_t - D\,u_{xx} = \nu\,[v(x,0^+,t)+v(x,0^-,t)] - 2\mu\,u & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ \mp d\,v_y(x,0^\pm,t) = \mu\,u - \nu\,v(x,0^\pm,t) & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \end{cases} \tag{3}$$

which corresponds to the case of a *plane* with a road of different diffusion. The factor 2 in the second equation of (3) takes into account the fact that the density u can pass to both sides of the surrounding field and gets positive contribution by the density v both from the upper and the lower part. These exchanges are compensated by the flux equations, i.e. the last relations in (3).

The main result that we provide for (3) is the following proposition, which, as it will be apparent in the proof of Proposition 7, will essentially be based on the study of the dependence of c^*_{hp} with respect to the exchange parameters μ and ν.

Proposition 1. *Problem* (3) *admits an asymptotic speed of propagation in the x direction, denoted by c^*_{p1} (referring to the planar field with one road), which satisfies:*

(i) $c^*_{\mathrm{p1}} \geq c_{\mathrm{KPP}}$*;*
(ii) $c^*_{\mathrm{p1}} > c_{\mathrm{KPP}}$ *if and only if* $D > 2d$*. In such a case,* $c^*_{\mathrm{p1}} < c^*_{\mathrm{hp}}$.

We observe that, the section of the field in (3) is unbounded as in (1) and there is a lower bound on the asymptotic speed of propagation in the direction of the road, given again by the classical Fisher-KPP speed. Indeed this is a general result which always holds true when the field has at least one component which is unbounded in every direction (see Lemma 1 below). Another point that (3) shares with (1) is that, when the diffusion in the field dominates - i.e. when $D \leq 2d$ -, the speed of propagation coincides with the one of the homogeneous case, while, when the diffusion on the road dominates, enhancement of the propagation speed takes place. Nevertheless, such an enhancement is reduced when the density is allowed to exchange on the two sides with respect to the case of one-side exchanges given by (1). This phenomenon is not a priori evident, since, despite the fact that in (3) the fraction of the density that leaves the line of fast diffusion is twice as much as in (1), also the contribution from the field doubles.

Finally, the last problem that we consider is

$$\begin{cases} v_t - d\,\Delta v = f(v) & \text{for } (x,y,t)\in\mathbb{R}\times\mathbb{R}\setminus\{\pm R\}\times\mathbb{R}^+, \\ u_t - D\,u_{xx} = \nu\,[v(x,\pm R^+,t)+v(x,\pm R^-,t)]-2\mu\,u & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ \mp d\,v_y(x,R^\pm,t) = \mu\,u - \nu\,v(x,R^\pm,t) & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ \mp d\,v_y(x,-R^\pm,t) = \mu\,u - \nu\,v(x,-R^\pm,t) & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \end{cases} \tag{4}$$

which describes a plane with *two* roads where the diffusion is different with respect to the one in the field, and for which the main result is the following.

Theorem 3. *Problem* (4) *admits an asymptotic speed of propagation in the x direction, denoted by c^*_{p2} (referring to the planar case with two roads), which satisfies:*

(i) $c^*_{\mathrm{p2}} \geq c_{\mathrm{KPP}}$*;*
(ii) $c^*_{\mathrm{p2}} > c_{\mathrm{KPP}}$ *if and only if $D > 2d$. In such a case, $R \mapsto c^*_{\mathrm{p2}}(R)$ is continuous, decreasing and satisfies*

$$\lim_{R\downarrow 0} c^*_{\mathrm{p2}}(R) = c^*_{\mathrm{hp}}, \qquad \lim_{R\uparrow\infty} c^*_{\mathrm{p2}}(R) = c^*_{\mathrm{p1}}. \tag{5}$$

In particular, $c^*_{\mathrm{p2}}(R) > c^*_{\mathrm{p1}}$ for all R;
(iii) if $D > 2d$, there exists $R^* \in (R_K, R_{\mathrm{hp}})$, where R_K and R_{hp} are the ones of Theorem 2, such that $c^*_{\mathrm{p2}}(R) > c^*_{\mathrm{st}}(R)$ if and only if $R < R^*$.

Once again, we see that an unbounded field in every direction makes the asymptotic speed of propagation bounded from below by c_{KPP}, with the usual threshold of the diffusion on the road in order to have enhancement. The main novelty here is the fact that, contrarily to the case of a strip bounded by two roads of fast diffusion, when such enhancing roads are placed in the whole plane and the distance between them increases, the speed of propagation always decreases. This means that the densities take advantage of the reaction in the field, no matter how it is distributed, and separating the roads of fast diffusion reduces their effect on the enhancement. In addition, observe that, when the roads enhance the speed of propagation, having two of them gives a better enhancement than in the case with only one road, as it is natural to expect. Finally, relations (5) can be seen once more as a continuous dependence of the speed of propagation with respect to the domain: when the strip between the roads shrinks, the effect inside it becomes negligible, as if the exchanges where one-sided; while, if the distance between the roads tends to infinity, considering one of them to be fixed makes the effect of the other one disappear.

The results of Theorems 1–3 are summarized in Figure 1.

This chapter is distributed as follows: in Section 2 we recall some preliminary results, from basic features of road-field systems up to the general way to construct the asymptotic speed of propagation; in Section 3 we consider the problems with one road, i.e. (1) and (3), we recall the proof of Theorem 1 given in [4], and we prove Proposition 1; finally, in Section 4, we consider the remaining problems, those with two roads, recalling the proof of Theorem 2 given in [19] and providing the one of Theorem 3, which is the main new result of this chapter.

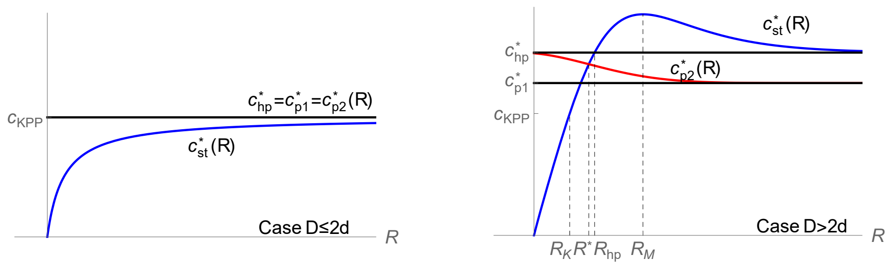

Fig. 1 Graphs of the asymptotic speed of propagation in the x direction for problems (1), (2), (3) and (4), considered as a function of R: (left) the case $D \leq 2d$ and (right) the case $D > 2d$.

2 Preliminary results: comparison principles, long-time behavior and existence of the asymptotic speed of propagation

In this section we recall some preliminary results that have been proved in [4] for system (1) and in [19] for system (2), and that can be easily adapted to the cases of systems (3) and (4). Without further mention, we stress that such results are valid for all the aforementioned systems, with the natural modifications due to the different domains in which they are posed. We begin with the following parabolic strong comparison principle.

Proposition 2 ([4]). *Let* $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *be, respectively, a subsolution bounded from above and a supersolution bounded from below of* (1) *such that* $(\underline{u},\underline{v}) \leq (\overline{u},\overline{v})$ *at* $t = 0$*, component-wise in their respective domains. Then,* $(\underline{u},\underline{v}) < (\overline{u},\overline{v})$ *for all* $t > 0$*, or there exists* $T > 0$ *such that* $(\underline{u},\underline{v}) = (\overline{u},\overline{v})$ *for all* $t < T$*.*

Then, we recall the well-posedness of the system, starting from nonnegative, bounded, continuous initial data (uniqueness, in particular, follows from Proposition 2).

Proposition 3 ([4]). *Let* (u_0, v_0) *be nonnegative, bounded and continuous. Then, there is a unique solution* (u, v) *satisfying* $\lim_{t\downarrow 0}(u, v) = (u_0, v_0)$.

The following is a comparison principle for a class of *generalized subsolutions*, that will be repeatedly used for the characterization of the asymptotic speed of propagation. Once again, we state it for system (1), although it is also valid, with the obvious modifications, for all the other systems.

Proposition 4 ([4]). *Let* (u_1, v_1) *be a subsolution of* (1) *bounded from above, and such that* u_1 *and* v_1 *vanish, respectively, on the boundary of an open set* E *of* $\mathbb{R}\times\ (0,+\infty)$*, and of an open set* F *of* $\{y > 0\} \times (0,+\infty)$ *(in the relative topologies). Assume in addition that*

$$\begin{aligned} v_1 &\leq 0 \qquad \text{in } \overline{E} \cap \{u_1 > 0\} \setminus \overline{F}, \\ u_1 &\leq 0 \qquad \text{in } \overline{F} \cap \{v_1 > 0\} \setminus \overline{E}. \end{aligned}$$

Then, setting

$$\underline{u} := \begin{cases} \max\{u_1, 0\} & \text{in } \overline{E}, \\ 0 & \text{otherwise,} \end{cases} \qquad \underline{v} := \begin{cases} \max\{v_1, 0\} & \text{in } \overline{F}, \\ 0 & \text{otherwise,} \end{cases}$$

for any supersolution $(\overline{u}, \overline{v})$ *of* (1) *bounded from below and such that* $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ *at* $t = 0$*, we have* $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ *for all* $t > 0$*.*

Next we present the result for the long-time behavior of the solutions. Although it is valid for all the systems (see [4, Section 4] for systems (1) and (3), and [19, Section 3] for system (2)), we state it for system (4) and give a sketch of the proof, which is slightly different from the other cases, since it requires the combination of the pieces of field with bounded and unbounded section.

Theorem 4. *Let* (u, v) *the solution of* (4) *with a nonnegative, continuous compactly supported initial datum* $(u_0, v_0) \neq (0, 0)$*. Then*

$$\lim_{t \to \infty} (u, v) = \left(\frac{\nu}{\mu}, 1 \right). \tag{6}$$

Proof. We proceed in several steps.

Step 1. The pair $\left(\frac{\nu}{\mu} K, K\right)$, where K is a sufficiently large constant, is a stationary supersolution of of (4) which lies above (u_0, v_0). Thus, the solution of (4) with this supersolution as an initial datum converges to a stationary solution of (4), denoted by (U_1, V_1), which, thanks to Proposition 2, satisfies

$$\limsup_{t \to \infty} (u, v) \leq (U_1, V_1). \tag{7}$$

In addition (U_1, V_1) is x independent and symmetric with respect to reflections about the x axis $\{y = 0\}$. Indeed, the solution of the parabolic problem and its limit as $t \to +\infty$ inherit the desired symmetries from the initial datum, from the fact that the Cauchy problems associated to (4) have a unique solution (thanks to Proposition 3), and that (4) is invariant by translations in x and by reflections about $\{y = 0\}$.

Step 2. By taking, for $\alpha, \beta, \varepsilon$ positive and small,

$$\underline{v} = \varepsilon \cos(\alpha x) \cos\left(\beta \left(y - R - 1 - \frac{\pi}{2\beta} \right) \right)$$

for $x \in \left(-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\right)$ and $y \in \left(R + 1, R + 1 + \frac{\pi}{\beta}\right)$, together with its reflection about $\{y = 0\}$ and extending to 0 elsewhere, we obtain a stationary generalized subsolution of (4) which lies below (u, v), the latter considered at $t = 1$. Proposition 4 thus gives the existence of a stationary solution (U_2, V_2) of (4) which is symmetric about the x axis and such that

$$(U_2, V_2) \leq \liminf_{t \to \infty} (u, v). \tag{8}$$

Finally, a sliding argument as the one of [5, Lemma 2.3] allows us to obtain the independence on x of (U_2,V_2).

Step 3. We claim that the unique nonnegative, bounded stationary solution of (4) which is x independent and symmetric about $\{y=0\}$ is $\left(\frac{\nu}{\mu},1\right)$. Thus, (7) and (8) allow us to obtain (6).

To prove the claim, consider a stationary solution $(U,V(y))$ with the mentioned symmetries. Thanks to the second equation in (4) for $y=R$, it satisfies

$$0=\nu\left(V(R^+)+V(R^-)\right)-2\mu U \tag{9}$$

and, thanks to the first one and the symmetry about $\{y=0\}$,

$$\begin{cases} -dV''(y)=f(V(y)), & y\in(0,R),\\ V'(0)=0. \end{cases}$$

We prove that $V(0)=1$, which, thanks to (KPP), will entail that $V\equiv 1$ and, as a consequence from (9), $U=\frac{\nu}{\mu}$. If, by contradiction, $V(0)\in(0,1)$, then (KPP) implies that V is concave and decreasing in $(0,R)$. By combining this with (9) and with the third equations in (4), we obtain

$$0>dV'(R^-)=\mu U-\nu V(R^-)=\nu V(R^+)-\mu U=dV'(R^+),$$

thus V would be decreasing and concave for all $y>R$, which is impossible, since it is positive. Similarly, we can exclude that $V(0)>1$, otherwise V would be convex and increasing for all $y\neq R$, thus unbounded. □

The following result, which relies on the comparison principles given in Propositions 2 and 4, will be used, together with the constructions performed in Sections 3 and 4, to obtain the existence of the asymptotic speed of propagation. Once again, we state it for system (4) even if it is valid, with the obvious due modifications, for all the road-field systems considered in this work. Since it is one of the core results, we also provide a sketch of the proof (for the details we refer to [5, 19]).

Proposition 5. *Assume that there exists $c^*>0$ such that:*

(i) for every $c\geq c^*$ there exist supersolutions of the linearized system around $(0,0)$

$$\begin{cases} v_t-d\Delta v=f'(0)v & \text{for } (x,y,t)\in\mathbb{R}\times\mathbb{R}\setminus\{\pm R\}\times\mathbb{R}^+,\\ u_t-Du_{xx}=\nu\left[v(x,\pm R^+,t)+v(x,\pm R^-,t)\right]-2\mu u & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ \mp d v_y(x,R^\pm,t)=\mu u-\nu v(x,R^\pm,t) & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ \mp d v_y(x,-R^\pm,t)=\mu u-\nu v(x,-R^\pm,t) & \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \end{cases} \tag{10}$$

of the form

$$(\overline{u},\overline{v})=e^{\pm\alpha(x\pm ct)}(1,\phi(y)), \tag{11}$$

where α is a positive constant, and $\phi(y)$ is positive in the field;

(ii) for all $c<c^*$, $c\sim c^*$ there exist arbitrarily small, nonnegative generalized stationary subsolutions $(\underline{u},\underline{v})$ of

$$\begin{cases} v_t - d\,\Delta v \pm c v_x = f(v) & \textit{for } (x,y,t) \in \mathbb{R}\times\mathbb{R}\setminus\{\pm R\}\times\mathbb{R}^+, \\ u_t - D u_{xx} \pm c u_x = \nu[v(x,\pm R^+,t) + v(x,\pm R^-,t)] - 2\mu u & \textit{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+, \\ \mp d\, v_y(x,R^\pm,t) = \mu\, u - \nu\, v(x,R^\pm,t) & \textit{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+, \\ \mp d\, v_y(x,-R^\pm,t) = \mu\, u - \nu\, v(x,-R^\pm,t) & \textit{for } (x,t) \in \mathbb{R}\times\mathbb{R}^+, \end{cases} \tag{12}$$

with $(\underline{u},\underline{v})$ *having compact support and being symmetric about* $\{y = 0\}$.[1]

Then c^* *is the asymptotic speed of propagation of problem* (4).

Proof. Take $k > 0$ sufficiently large so that $k(\overline{u},\overline{v})$, where $(\overline{u},\overline{v})$ is the supersolution given by assumption (i) for $c = c^*$ with the "$-$" sign, lies above (u_0, v_0). Observe that, since system (10) is linear, $k(\overline{u},\overline{v})$ is still a supersolution of (10) and thus, thanks to (KPP), it is a supersolution to (4).

Consider $c > c^*$ and $x > ct$; then, thanks to Proposition 2,

$$(u,v) < ke^{-\alpha(x-c^*t)}(1,\phi(y)) < ke^{\alpha(c^*-c)t}(1,\phi(y)) \to 0$$

as $t \to \infty$, proving the first part of the definition of asymptotic speed of propagation for the propagation to the right. For the propagation to the left we reason similarly, by taking the supersolution in (i) with the "$+$" sign.

On the other hand, using the subsolutions given by assumption (ii), one can prove, following the same lines of Theorem 4, that, for $c < c^*$, with c arbitrarily close to c^*,

$$\lim_{t\to\infty}(u(x \pm ct, t), v(x \pm ct, y, t)) = \left(\frac{\nu}{\mu}, 1\right),$$

and then the second part of the definition of asymptotic speed of propagation follows by applying [5, Lemma 4.1] (see also [19, Lemma 4.4] for a proof of it). □

3 Characterization of the asymptotic speed of propagation for problems with one road

This section is devoted to the construction and a geometric characterization of the asymptotic speed of propagation for the road-field systems considered in the introduction having one road of different diffusion, i.e. problems (1) and (3).

The following lemma, whose proof is based on [4, Lemma 6.2], constructs subsolutions with the characteristics of assumption (ii) of Proposition 5 (with (12) adequately replaced in each case by the corresponding parabolic problem with additional transport terms $\pm cv_x$, $\pm cu_x$), when $D \le 2d$, $0 < c < c_{\text{KPP}}$, and the field has at least one component whose section is unbounded in y, i.e. for systems (1),(3) and (4). Proposition 5 will thus entail that the speed of propagation for these three systems is larger than or equal to c_{KPP}.

---

[1] This symmetry condition is not required - and even meaningless - when the domain is the upper half-plane.

Lemma 1. *Let $D \le 2d$ and $0 < c < c_{\mathrm{KPP}}$. Then, there exist arbitrarily small, nonnegative generalized stationary subsolutions of* (12) *(and the analogous versions corresponding to systems* (1) *and* (3)*) with compact support and symmetric about $\{y = 0\}$ (when the domain has this symmetry too).*

Proof. We look for subsolutions of the form

$$\underline{v} = \varepsilon \psi(x) \cos\left(\beta\left(y - R - 1 - \frac{\pi}{2\beta}\right)\right) \tag{13}$$

for β, ε positive and small, $y \in \left(R+1, R+1+\frac{\pi}{\beta}\right)$, and $\psi(x)$ nonnegative with compact support to be determined. We also take the reflection of $\underline{v}$ about $\{y = 0\}$ and extend to 0 elsewhere in the field.

Observe that, if $\underline{v}$ is small enough (i.e., if ε is small enough), solves

$$-d\Delta v \pm c v_x = (f'(0) - \delta) v \tag{14}$$

for $\delta \in (0, f'(0))$, $\delta \sim 0$, and its support is contained in the field, then, by taking $\underline{u} = 0$, we obtain a subsolution to (12). For (13) to solve (14), $\psi(x)$ has to satisfy

$$d\psi'' \mp c\psi' + (f'(0) - \delta - d\beta^2)\psi = 0,$$

thus, since $0 < c < c_{\mathrm{KPP}}$, for $\delta, \beta \sim 0$ $\psi(x)$ is given by $e^{\lambda x}$, where $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a root of the associated characteristic polynomial. In order to obtain a real solution, we take the real part of ψ and, to have compact support in x, we take only one oscillation and extend to 0 elsewhere. □

The following proposition, which summarizes the content of [4, Sections 5–6], gives a geometrical characterization of c^*_{hp} and, combined with Proposition 5, allows us to prove Theorem 1. We recall the elements of its proof, since they will be used also in the rest of this chapter.

Proposition 6 ([4]). *Problem* (1) *admits an asymptotic speed of propagation c^*_{hp} in the x direction which, for $D \le 2d$, satisfies $c^*_{\mathrm{hp}} = c_{\mathrm{KPP}}$, while, for $D > 2d$, it satisfies $c^*_{\mathrm{hp}} > c_{\mathrm{KPP}}$ and is the smallest value of c for which the curves*

$$\alpha^-_{d,\mathrm{hp}}(c,\beta) := \frac{c - \sqrt{c^2 - c^2_{\mathrm{KPP}} - 4d^2\beta^2}}{2d}, \quad \alpha^+_{D,\mathrm{hp}}(c,\beta) := \frac{c + \sqrt{c^2 + \frac{4\cdot\mu\cdot Dd\beta}{\nu + d\beta}}}{2D} \tag{15}$$

have real intersections.

Proof. Let us begin with the case $D \le 2d$. Thanks to Lemma 1, in order to apply Proposition 5, it is sufficient to construct, for every $c \ge c_{\mathrm{KPP}}$ supersolutions of the linearization of (1) around $(0,0)$ of the form (11). To this end, we take $\phi(y) = \gamma e^{-\beta y}$, with $\beta \ge 0$, $\gamma > 0$ and, by plugging into the linearized system, we obtain that such a candidate is a solution (respectively, a supersolution) if and only if the following algebraic system, involving the unknowns α, β, γ and the parameter c,

$$\begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \nu\gamma - \mu \\ d\beta\gamma = \mu - \nu\gamma \end{cases} \iff \begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \frac{-\mu d\beta}{\nu + d\beta} \\ \gamma = \frac{\mu}{\nu + d\beta} \end{cases} \tag{16}$$

is satisfied (respectively, if and only if it is satisfied with the equality signs replaced by "$\geq$").

The first equation in (16) describes, for $c \geq c_{\text{KPP}}$, a circle $\Sigma_d(c)$ in the (β, α) plane with center $\left(0, \frac{c}{2d}\right)$ and radius $\frac{\sqrt{c^2 - c_{\text{KPP}}^2}}{2d}$. Observe that $\Sigma_d(c)$ degenerates to its center as $c \downarrow c_{\text{KPP}}$.

When $D \leq 2d$ and $c \geq c_{\text{KPP}}$, by taking $(\alpha, \beta, \gamma) = \left(\frac{c}{2d}, 0, \frac{\mu}{\nu}\right)$, which amounts to consider the center of Σ_d in the (β, α) plane, the relations in (16) are satisfied with "$\geq$", and we have constructed the desired supersolution.

To treat the case $D > 2d$, we explicitly write the curve given by the second relation of (16) as a function of β and the parameter c, obtaining the curve $\alpha^+_{D,\text{hp}}(c, \beta)$ defined in (15) - we only consider the branch with the "+" in front of the square root, since this will be enough for the construction, as it will be apparent from the following discussion. We observe that such a curve intersects the α-axis in the point $\left(\frac{c}{D}, 0\right)$; thus, since $D > 2d$, the circle Σ_d arises, for $c = c_{\text{KPP}}$, above $\alpha^+_{D,\text{hp}}$. In addition, the lower part of the circle, which is parameterized by the function $\alpha^-_{d,\text{hp}}(c, \beta)$, introduced in (15) as well, is decreasing with respect to c and converges to 0 as $c \to \infty$, while $\alpha^+_{D,\text{hp}}(c, \beta)$ is increasing in c and tends to ∞ as $c \to \infty$. Therefore, since these curves are regular, there exists a least value of c, denoted by c^*_{hp}, which is greater than c_{KPP} and for which they intersect for the first time, being tangent, and they intersect strictly for every greater c.

To conclude the proof, we show, thanks to Proposition 5, that c^*_{hp} is the asymptotic speed of propagation. By construction, there are solutions of (16) for every $c \geq c^*_{\text{hp}}$, providing solutions of the linearized system.

To construct compactly supported subsolutions for $c < c^*_{\text{hp}}$, $c \sim c^*_{\text{hp}}$, consider the truncation of problem (1) obtained by considering $0 < y < L$ and imposing $v(x, L, t) = 0$. Reasoning as above, i.e. studying the corresponding system for α, β, γ, it is possible to construct solutions of the linearized truncated system with penalization, i.e. with $f'(0)$ replaced by $f'(0) - \delta$, of type $e^{\pm\alpha(x \pm ct)}(1, \gamma \sinh(\beta(L - y)))$ for c greater than or equal to a certain value $c^*(L, \delta) < c^*_{\text{hp}}$. Moreover, for c smaller than $c^*(L, \delta)$, arbitrarily close to it, it is possible to show by using Rouché's theorem (see [4, Lemma 6.1]) that the system for α, β, γ has complex solutions, giving complex solutions of the linearized truncated system. Taking the real part of such solutions, which oscillates in x, considering only one oscillation - as in the proof of Lemma 1 -, extending to 0 and taking small multiples, gives a compactly supported subsolution to the original problem. Since $\lim_{(L,\delta) \to (\infty, 0)} c^*(L, \delta) = c^*_{\text{hp}}$, this procedure allows us to construct subsolutions satisfying assumption (ii) of Proposition 5 for $c < c^*_{\text{hp}}$, arbitrarily close to it, which concludes the proof. □

This geometric characterization allows us to prove almost immediately Theorem 1, for which we recall once more the elements of the proof given in [4].

Proof (of Theorem 1). The existence of $c^*_{\rm hp}$ and parts (i) and (ii) are contained in Proposition 6.

Passing to (iii), we observe that if, by contradiction, $c^*_{\rm hp}(D)$ was bounded, the second curve in (15) would converge locally uniformly to 0 as $D \to \infty$, thus it would not have any intersection with the first one, against the characterization of $c^*_{\rm hp}(D)$ given in Proposition 6. □

This completes the review of the results for the half-plane with one road and we pass now to construct the speed of propagation for the case of a plane with one road (3), which is the content of the following proposition. Then, we give the proof of Proposition 1.

Proposition 7. *Problem* (3) *admits an asymptotic speed of propagation $c^*_{\rm p1}$ in the x direction which, for $D \le 2d$, satisfies $c^*_{\rm p1} = c_{\rm KPP}$, while, for $D > 2d$, it satisfies $c^*_{\rm p1} > c_{\rm KPP}$ and is the smallest value of c for which the curves*

$$\alpha^-_{d,\rm p1}(c,\beta) := \frac{c - \sqrt{c^2 - c^2_{\rm KPP} - 4d^2\beta^2}}{2d}, \quad \alpha^+_{D,\rm p1}(c,\beta) := \frac{c + \sqrt{c^2 + \frac{4\cdot 2\mu\cdot Dd\beta}{\nu + d\beta}}}{2D} \tag{17}$$

have real intersections.

Proof. The construction follows the same lines of the one of Proposition 6 and relies on Proposition 5. On the one hand, one looks for supersolutions of type $\overline{u} = e^{\pm\alpha(x\pm ct)}$, $\overline{v} = \gamma e^{\pm\alpha(x\pm ct)-\beta y}$ for $y > 0$ and $\overline{v} = \gamma e^{\pm\alpha(x\pm ct)+\beta y}$ for $y < 0$, and obtains the algebraic system

$$\begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = 2\nu\gamma - 2\mu \\ d\beta\gamma = \mu - \nu\gamma \end{cases} \iff \begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \frac{-2\mu d\beta}{\nu + d\beta} \\ \gamma = \frac{\mu}{\nu + d\beta}, \end{cases} \tag{18}$$

which leads, when $D > 2d$, to the search for real intersections between the curves (17).

On the other hand, the construction of compactly supported subsolutions for $c < c^*_{\rm p1}$, $c \sim c^*_{\rm p1}$, follows, in the case $D \le 2d$, from Lemma 1 and, in the case $D > 2d$, by truncating in y and using Rouché's theorem to obtain complex solutions, exactly as indicated in the proof of Proposition 6. □

Proof (of Proposition 1). The existence of $c^*_{\rm p1}$, its lower bound and the threshold for enhancement with respect to $c_{\rm KPP}$ have already been proved in Proposition 7.

It only remains to show that $c^*_{\rm p1} < c^*_{\rm hp}$ when $D > 2d$, and, for this, it is sufficient to observe that $\alpha^-_{d,\rm p1}(c,\beta)$ in (17) coincides with $\alpha^-_{d,\rm hp}(c,\beta)$ in (15), while $\alpha^+_{D,\rm p1}(c,\beta)$ is obtained from $\alpha^+_{D,\rm hp}(c,\beta)$ by replacing μ with 2μ. As a consequence, thanks to the geometric characterization given in Propositions 6 and 7, if we explicitly point out the dependence of $c^*_{\rm hp}$ and $c^*_{\rm p1}$ with respect to the parameter μ, we have that, always for $D > 2d$, $c^*_{\rm p1}(\mu) = c^*_{\rm hp}(2\mu)$, and, in order to get the conclusion, it is sufficient to show that $\mu \mapsto c^*_{\rm hp}(\mu)$ is decreasing. To this end, observe that the function $\mu \mapsto \alpha^+_{D,\mathrm{hp}}(c,\beta,\mu)$ is strictly increasing and, by construction, the curves $\alpha^+_{D,\mathrm{hp}}(c^*_{\mathrm{hp}}(\mu),\beta,\mu)$ and $\alpha^-_{d,\mathrm{hp}}(c^*_{\mathrm{hp}}(\mu),\beta)$ are tangent for every μ. Thus, if $\mu_1 < \mu_2$, $\mu_1 \sim \mu_2$, the curves $\alpha^+_{D,\mathrm{hp}}(c,\beta,\mu_2)$ and $\alpha^-_{d,\mathrm{hp}}(c,\beta)$ are strictly secant for $c = c^*_{\mathrm{hp}}(\mu_1)$ and, due to the monotonicities in c, this parameter has to be decreased in order to obtain the value for which they intersect for the first time, which, thanks again to Proposition 6, provides us with $c^*_{\mathrm{hp}}(\mu_2)$. □

4 Characterization of the asymptotic speed of propagation for problems with two roads

In this section we construct the asymptotic speed of propagation for the two remaining problems (2) and (4), those with two roads. We preliminarily observe that such problems are symmetric with respect to reflections about $\{y=0\}$. For this reason, we will construct the super- and subsolutions needed to apply Proposition 5 with the same symmetry, i.e. we will look for functions defined only on $\{y>0\}$ and satisfying $v_y(0^+)=0$, and then will consider their even extension on $\{y<0\}$.

We begin with problem (2) for a strip-shaped field $\{y \in (-R,R)\}$. In this case, the construction of the speed of propagation is more complicated than in the cases presented in Section 3, since the eigenvalue problem $-\phi''(y) = \lambda\phi(y)$ has two types of positive eigenfunctions in $(-R,R)$ which satisfy $\phi'(0)$=0: $\cos(\sqrt{\lambda}y)$ for $\lambda \in \left(0, \frac{\pi}{2R}\right)$, and $\cosh(\sqrt{-\lambda}y)$ for $\lambda < 0$. This entails that we have to consider two types of supersolutions of the form (11): the first type with $\overline{v}_1 = \gamma e^{\pm\alpha(x\pm ct)}\cos(\beta y)$, $0<\beta<\frac{\pi}{2R}$, and the second one $\overline{v}_2 = \gamma e^{\pm\alpha(x\pm ct)}\cosh(\beta y)$. The geometric characterization of the asymptotic speed of propagation that we obtain in this case is the following one, and the proof we provide summarizes the results of [19, Section 4].

Proposition 8 ([19]). *Problem* (2) *admits an asymptotic speed of propagation c^*_{st} in the x direction which is the smallest value of the parameter c for which either the curves,*

$$\alpha^{\pm}_{d,\mathrm{st},1} := \frac{c \pm \sqrt{c^2 - c^2_{\mathrm{KPP}} + 4d^2\beta^2}}{2d}, \quad \alpha^{\pm}_{D,\mathrm{st},1} := \frac{c \pm \sqrt{c^2 - \frac{4\mu D d\beta \sin(\beta R)}{\nu\cos(\beta R) - d\beta\sin(\beta R)}}}{2D}, \tag{19}$$

or

$$\alpha^{-}_{d,\mathrm{st},2} := \frac{c - \sqrt{c^2 - c^2_{\mathrm{KPP}} - 4d^2\beta^2}}{2d}, \quad \alpha^{+}_{D,\mathrm{st},2} := \frac{c + \sqrt{c^2 + \frac{4\mu D d\beta \sinh(\beta R)}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)}}}{2D} \tag{20}$$

have real intersections in the first quadrant of the (β,α) plane (in (19) *we consider $0<\beta<\overline{\beta}<\frac{\pi}{2R}$, where $\overline{\beta}$ is the first zero of the denominator inside the square root of $\alpha^{\pm}_{D,\mathrm{st},1}$).*

If intersection first occurs between the curves (19)*, then* $c^*_{\rm st}$ *is said to be of* type 1*, and will be denoted by* $c^*_{{\rm st},1}$*, otherwise, if intersection first occurs between the curves in* (20)*, then we say that* $c^*_{\rm st}$ *is of* type 2 *and we denote it by* $c^*_{{\rm st},2}$.

Proof. The proof follows similar lines as the ones of Section 3: to construct the above-mentioned supersolutions of type 1, one has to find solutions of the following system

$$\begin{cases} c\alpha - d\alpha^2 + d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \nu\gamma\cos(\beta R) - \mu \\ -d\beta\gamma\sin(\beta R) = \mu - \nu\gamma\cos(\beta R) \end{cases} \iff \begin{cases} c\alpha - d\alpha^2 + d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \frac{\mu d\beta\sin(\beta R)}{\nu\cos(\beta R) - d\beta\sin(\beta R)} \\ \gamma = \frac{\mu}{\nu\cos(\beta R) - d\beta\sin(\beta R)} \end{cases} \tag{21}$$

(observe that, for $0 < \beta < \overline{\beta}$, $\gamma > 0$); while for supersolutions of type 2, one reduces to system

$$\begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \nu\gamma\cosh(\beta R) - \mu \\ d\beta\gamma\sinh(\beta R) = \mu - \nu\gamma\cosh(\beta R) \end{cases} \iff \begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \frac{-\mu d\beta\sinh(\beta R)}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)} \\ \gamma = \frac{\mu}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)}. \end{cases} \tag{22}$$

System (21) leads to find intersections between the curves in (19), while (22) between those in (20).

In order to conclude, assume that $c^*_{\rm st}$ is of type 1. Then, it is possible to reason as in the proof of Proposition 5 to show that for $c < c^*_{\rm st}$, $c \sim c^*_{\rm st}$, system (21) admits complex solutions which can be used to construct the desired compactly supported subsolutions. The same can be done by using system (22) when $c^*_{\rm st}$ is of type 2. We remark that no truncation is needed here to obtain a compact support in y, since y is already bounded. □

Remark 1. By studying the dependence on c of the curves (19) and (20), one can observe that in both cases they have real intersections for sufficiently large c, thus both provide us with supersolutions of the problem. As a consequence, one might think that two different values for $c^*_{\rm st}$ can be obtained, one for each pair of curves. Nonetheless, the analysis of [19] (see in particular Section 4 and Proposition 4.1) guarantees that the construction of compactly supported subsolutions only works in one of the two cases, entailing in particular that the definition of the type of $c^*_{\rm st}$ given in Proposition 8 is well posed.

Proof (of Theorem 2). The existence follows from Proposition 8. Moreover, it is possible to show (see [19, Section 4]) that

$$\text{if } D \le 2d, \quad c^*_{\rm st} = c^*_{{\rm st},1} \text{ for all } R > 0, \tag{23}$$

$$\text{if } D > 2d, \quad c^*_{\rm st} = \begin{cases} c^*_{{\rm st},1} \text{ for } R \in (0, R_M), \\ c^*_{{\rm st},2} \text{ for } R > R_M, \end{cases} \tag{24}$$

where we use the notation introduced in Proposition 8. We are now ready to prove the qualitative properties of $c^*_{\rm st}$.

(i) For $R \sim 0$, (23) and (24) give that $c^*_{\text{st}} = c^*_{\text{st},1}$. In addition, as $R \downarrow 0$, the curve $\alpha^{\pm}_{D,\text{st},1}$ converges to the horizontal lines $\alpha = 0$ and $\alpha = c/D$. Thus, for any fixed c, there are always intersections between such a curve and $\alpha^{\pm}_{d,\text{st},1}$, which connects, in the first quadrant of the (β,α) plane, the points $\left(\sqrt{\frac{f'(0)}{d}}, 0\right)$ and (∞,∞). Proposition 8 therefore gives that $\lim_{R\downarrow 0} c^*_{\text{st}}(R) = 0$.

(ii) We distinguish the cases $D \leq 2d$ and $D > 2d$. In the former one, thanks to (23), we only have to consider system (21), and we observe that, when $c = c_{\text{KPP}}$, if we take $(\alpha,\beta,\gamma) = \left(\frac{c_{\text{KPP}}}{2d}, 0, \frac{\mu}{\nu}\right)$, the first and third relation of such a system hold true, while the second one holds true with the "$\geq$" sign. Thus, $c^*_{\text{st}}(R) < c_{\text{KPP}}$ for all R and $\limsup_{R\to\infty} c^*_{\text{st}}(R) \leq c_{\text{KPP}} = c^*_{\text{hp}}$. On the other hand, observe that the construction of compactly supported subsolutions of Lemma 1 can be carried out for sufficiently large R, entailing that $\liminf_{R\to\infty} c^*_{\text{st}}(R) \geq c_{\text{KPP}}$, which concludes the proof in this case.
When $D > 2d$, according to (24), c^*_{st} is of type 2 for sufficiently large R. Now, the convergence of $c^*_{\text{st}}(R)$ to c^*_{hp} as $R \to \infty$ follows from the geometric characterizations given in Propositions 6 and 8, observing that $\alpha^{+}_{D,\text{st},2}$ converges, as $R \to \infty$, to $\alpha^{+}_{D,\text{hp}}$ locally uniformly in β.

(iii) Once again, if $D \leq 2d$, (23) guarantees that $c^*_{\text{st}} = c^*_{\text{st},1}$ for all R. One then proves that the curves $\alpha^{\pm}_{D,\text{st},1}$ shrink continuously as R increases, while the curves $\alpha^{\pm}_{d,\text{st},1}$ do not depend on R, entailing that $c^*_{\text{st},1}$ is continuous and increasing.

(iv) To prove that, if $D > 2d$, $c^*_{\text{st}}(R)$ is increasing for $R \in (0, R_M)$, we use (24) and reason as in the previous point. Similarly, we use (24) and the fact that $\alpha^{+}_{D,\text{st},2}$ increases, with respect to R, to obtain that $c^*_{\text{st}} = c^*_{\text{st},2}$ is decreasing for $R > R_M$. The continuity of the function $R \mapsto c^*_{\text{st}}$ is obvious for $R \neq R_M$, since the curves in (19) and (20) depend continuously on R. For $R = R_M$, the conclusion is not direct, since a transition of type occurs. Nevertheless, one proves that $\alpha^{+}_{d,\text{st},1}$ and $\alpha^{-}_{D,\text{st},1}$ do not play a role for $R = R_M$, and observes that, for $\beta = 0$, the remaining curves in (19) and those in (20) match in a differentiable way, which allows us to obtain the continuity of $c^*_{\text{st}}(R)$ also for $R = R_M$.
The existence and properties of R_{hp} and R_K now follow directly form the continuity and monotonicity properties of c^*_{st}, together with properties (i) and (ii) of Theorem 2 and (ii) of Theorem 1. □

Finally, we pass to the case of the plane with two roads (4), for which, as already remarked, Lemma 1 entails that $c^*_{\text{p2}} \geq c_{\text{KPP}}$.

Differently from the case of the strip, here it is enough to consider only one type of supersolution. Indeed, the unique *positive* eigenfunctions of $-\phi''(y) = \lambda\phi(y)$ for $|y| > R$ are the ones of exponential type. The differential equation in the field being the same for $|y| < R$, this forces to take $\overline{v}(x,y,t) = \gamma_1 e^{\pm\alpha(x\pm ct)}\cosh(\beta y)$ in $(-R,R)$ (recall that, by symmetry, we consider functions satisfying $\overline{v}_y(x,0^+,t) = 0$), excluding in this way the cosine. Moreover, in analogy with the constructions of Section 3, we take $\overline{u}(x,t) = e^{\pm\alpha(x\pm ct)}$ and $\overline{v}(x,y,t) = \gamma_2 e^{\pm\alpha(x\pm ct)-\beta(y-R)}$ for $y > R$. As usual, the constants $\alpha,\beta,\gamma_1,\gamma_2$ will be sought to be positive.

After these preliminaries, we show in the following proposition that these supersolutions suffice for the construction and characterization of the speed of propagation for this problem.

Proposition 9. *Problem* (4) *admits an asymptotic speed of propagation in the x direction c^*_{p2}, which, for $D \le 2d$, satisfies $c^*_{\mathrm{p2}} = c_{\mathrm{KPP}}$, while, for $D > 2d$, it satisfies $c^*_{\mathrm{p2}} > c_{\mathrm{KPP}}$ and is the smallest value of c for which the curves*

$$\begin{aligned} \alpha^-_{d,\mathrm{p2}}(c,\beta) &:= \frac{c - \sqrt{c^2 - c^2_{\mathrm{KPP}} - 4d^2\beta^2}}{2d}, \\ \alpha^+_{D,\mathrm{p2}}(c,\beta) &:= \frac{c + \sqrt{c^2 + 4\mu D\left(\frac{d\beta}{\nu + d\beta} + \frac{d\beta\sinh(\beta R)}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)}\right)}}{2D} \end{aligned} \tag{25}$$

have real intersections.

Proof. By plugging the above described candidate to supersolution into the linearization, the system that we obtain in this case reads

$$\begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = \nu(\gamma_1\cosh(\beta R) + \gamma_2) - 2\mu \\ d\beta\gamma_1\sinh(\beta R) = \mu - \nu\gamma_1\cosh(\beta R) \\ d\beta\gamma_2 = \mu - \nu\gamma_2 \end{cases} \iff$$

$$\iff \begin{cases} c\alpha - d\alpha^2 - d\beta^2 = f'(0) \\ c\alpha - D\alpha^2 = -\frac{\mu d\beta}{\nu + d\beta} - \frac{\mu d\beta\sinh(\beta R)}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)} \\ \gamma_1 = \frac{\mu}{\nu\cosh(\beta R) + d\beta\sinh(\beta R)} \\ \gamma_2 = \frac{\mu}{\nu + d\beta}. \end{cases} \tag{26}$$

As in Section 3, if $D \le 2d$, $(\alpha,\beta,\gamma_1,\gamma_2) = \left(\frac{c}{2d}, 0, \frac{\mu}{\nu}, \frac{\mu}{\nu}\right)$ provides us with supersolutions to (26) for $c \ge c_{\mathrm{KPP}}$. Thus, Proposition 5 and Lemma 1 allow us to conclude that $c^*_{\mathrm{p2}} = c_{\mathrm{KPP}}$ in this case. When $D > 2d$, instead, intersections between the curves in (25) provide us with solutions to (26) and, as a consequence, supersolutions to (4). Thanks to the monotonicity with respect to c, intersections between such curves exist for c greater than or equal to a certain value (greater than c_{KPP}) for which, as usual, the curves are tangent, and which will turn out to be c^*_{p2}.

Indeed, in order to apply Proposition 5, we only have to obtain compactly supported subsolutions in the case $D > 2d$ for $c < c^*_{\mathrm{p2}}$, $c \sim c^*_{\mathrm{p2}}$. To do so, we proceed as in the proof of Proposition 6: we consider the truncation at $y = L$, with $L \gg R$, by imposing $v(x,L,t) = 0$, and, by using Rouché's theorem, we prove that, for $c < c^*_{\mathrm{p2}}$, $c \sim c^*_{\mathrm{p2}}$, the associated linearized system with penalization admits complex solutions which allow us to obtain subsolutions whose support is compact also in the x variable. □

Proof (of Theorem 3). The existence part, (i) and the first part of (ii) are contained in Proposition 9. In (ii), it remains to prove the behavior with respect to R in the case $D > 2d$: the continuity and monotonicity follow since the map $R \mapsto \alpha^+_{D,\mathrm{p2}}$ is continuous and increasing, while $\alpha^-_{d,\mathrm{p2}}$ does not depend on R. Thus, since the curves in (25) are tangent for $c = c^*_{\mathrm{p2}}(R)$, if $R' > R$, $R' \sim R$, they are strictly secant and c has to be reduced in order to obtain the tangency situation.

The curve $\alpha^+_{D,\mathrm{p2}}$ converges locally uniformly to $\alpha^+_{D,\mathrm{hp}}$ as $R \downarrow 0$, and to $\alpha^+_{D,\mathrm{p1}}$ as $R \to \infty$. This proves the limits in (5). The fact that $c^*_{\mathrm{p2}}(R) > c^*_{\mathrm{p1}}$ follows from the monotonicity of $c^*_{\mathrm{p2}}(R)$ and the second limit in (5).

Passing to part (iii), by the continuity of $c^*_{\mathrm{st}}(R)$ and the properties of R_{hp}, we have $c^*_{\mathrm{st}}(R_{\mathrm{hp}}) = c^*_{\mathrm{hp}} > c^*_{\mathrm{p2}}(R_{\mathrm{hp}})$, where the last inequality follows from (ii) here. On the other hand, the properties of R_K, together with (ii) of Proposition 1 and (ii) here, give $c^*_{\mathrm{st}}(R_K) = c_{\mathrm{KPP}} < c^*_{\mathrm{p1}} < c^*_{\mathrm{p2}}(R_K)$. The existence of R^* and its properties now follow by the continuity and monotonicities of $c^*_{\mathrm{st}}(R)$ and $c^*_{\mathrm{p2}}(R)$. □

Acknowledgements This work has been supported by the Spanish Ministry of Economy, Industry and Competitiveness through contract Juan de la Cierva Incorporación IJCI-2015-25084 and project MTM2015-65899-P, and by the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi "Reaction-Diffusion Equations, Propagation and Modelling".

The author wishes to thank the anonymous referee for his/her comments which have improved the presentation of the results of this work.

References

1. D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), 33–76.
2. H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre, L. Rossi, Speed-up of reaction-diffusion fronts by a line of fast diffusion, in *Séminaire Laurent Schwartz—Équations aux Dérivées Partielles et Applications. Année 2013–2014*, Exp. No. XIX, 25 pp., Ed. Éc. Polytech., Palaiseau, 2014.
3. H. Berestycki, A.-C. Coulon, J.-M. Roquejoffre, L. Rossi, *The effect of a line with nonlocal diffusion on Fisher-KPP propagation*, Math. Models Methods Appl. Sci. **25** (2015), 2519–2562.
4. H. Berestycki, J.-M. Roquejoffre, L. Rossi, *The influence of a line with fast diffusion on Fisher-KPP propagation*, J. Math. Biol. **66** (2013), 743–766.
5. H. Berestycki, J.-M. Roquejoffre, L. Rossi, *Fisher-KPP propagation in the presence of a line: further effects*, Nonlinearity **26** (2013), 2623–2640.
6. H. Berestycki, J.-M. Roquejoffre, L. Rossi, *The shape of expansion induced by a line with fast diffusion in Fisher-KPP equations*, Comm. Math. Phys. **343** (2016), 207–232.
7. H. Berestycki, J.-M. Roquejoffre, L. Rossi, *Travelling waves, spreading and extinction for Fisher-KPP propagation driven by a line with fast diffusion*, Nonlinear Anal. **137** (2016), 171–189.
8. L. Dietrich, *Existence of travelling waves for a reaction–diffusion system with a line of fast diffusion*, Appl. Math. Res. Express. AMRX (2015), 204–252.
9. L. Dietrich, *Velocity enhancement of reaction-diffusion fronts by a line of fast diffusion*, Trans. Amer. Math. Soc. **369** (2017), 3221–3252.
10. L. Dietrich, J.-M. Roquejoffre, *Front propagation directed by a line of fast diffusion: large diffusion and large time asymptotics*, J. Éc. polytech. Math. **4** (2017), 141–176.

11. R. Ducasse, *Influence of the geometry on a field?road model: the case of a conical field*, J. London Math. Soc., doi:10.1112/jlms.12114.
12. N.R. Faria et al., *The early spread and epidemic ignition of HIV-1 in human populations*, Science **346** (2014), 56–61.
13. R.A. Fisher, *The wave of advantage of advantageous genes*, Annals of Eugenics **7** (1937), 355–369.
14. T. Giletti, L. Monsaingeon, M. Zhou, *A KPP road-field system with spatially periodic exchange terms*, Nonlinear Anal. **128** (2015), 273–302.
15. A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bull. Univ. État Moscou, Sér. Intern. A **1** (1937), 1–26.
16. A. Pauthier, *Uniform dynamics for Fisher-KPP propagation driven by a line of fast diffusion under a singular limit*, Nonlinearity **28** (2015), 3891–3920.
17. A. Pauthier, *The influence of nonlocal exchange terms on Fisher-KPP propagation driven by a line of fast diffusion*, Commun. Math. Sci. **14** (2016), 535–570.
18. C. Robinet, C. Suppo, E. Darrouzet, *Rapid spread of the invasive yellow-legged hornet in France: the role of human-mediated dispersal and the effects of control measures*, J. Appl. Ecol. **54** (2017), 205–215.
19. L. Rossi, A. Tellini, E. Valdinoci, *The effect on Fisher-KPP propagation in a cylinder with fast diffusion on the boundary*, SIAM J. Math. Anal. **49** (2017), 4595–4624.
20. A. Tellini, *Propagation speed in a strip bounded by a line with different diffusion*, J. Differential Equations **260** (2016), 5956–5986.